**2958. Proposed by R. P. BAKER, University of Iowa.**

Over a frictionless pulley a weightless cord sustains at one end a mass $M$, while the other end is wound on the axle of a wheel of mass $M$ and moment of inertia $N$. At the zero of time the wheel revolves with angular velocity $\omega$ and tends to wind up the cord. Describe the motion neglecting friction.

## SOLUTIONS

**2813 [1920, 81]. Proposed by PAUL CAPRON, U. S. Naval Academy.**

An ellipse having the major-axis $2a$ and the eccentricity $\epsilon$, is revolved first about its major axis, forming a prolate spheroid, then about its minor axis forming an oblate spheroid. Show that the surfaces of these spheroids are, respectively,

$$2\pi a^2(1/\epsilon)(\sqrt{1-\epsilon^2}\sin^{-1}\epsilon + 1)$$

and

$$2\pi a^2\left[2 + (1/\epsilon)(1-\epsilon^2)\log\left(\frac{1+\epsilon}{1-\epsilon}\right)\right].$$

### SOLUTION BY H. S. UHLER, Yale University.

*Case I.* Prolate spheroid.

Let the equation of the ellipse be

$$b^2x^2 + a^2y^2 - a^2b^2 = 0. \tag{1}$$

We may take as element of surface the lateral area of the frustum of a right circular cone having the following specifications: planes of bases normal to the $x$-axis and at distances $x$ and $x + dx$ from the origin, and generatrix of lateral surface tangent to the revolving ellipse at the point: $(x, y)$. Slant height $\equiv dt$.

Then

$$dt = dx\sqrt{1 + (dy/dx)^2}.$$

From (1)

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y};$$

hence

$$dt = \frac{bdx}{a^2y}\sqrt{a^4 - (a^2-b^2)x^2}.$$

Accordingly, the element of surface, $d\sigma$, is given by

$$d\sigma = 2\pi y dt = \frac{2\pi b dx}{a^2}\sqrt{a^4-(a^2-b^2)x^2},$$

and the required area of the prolate spheroid,

$$A_1 = \frac{4\pi b}{a^2}\int_0^a dx\sqrt{a^4-(a^2-b^2)x^2}. \tag{2}$$

By making use of the standard formula,

$$\int d\varphi\sqrt{n^2-\varphi^2} = \frac{1}{2}\varphi\sqrt{n^2-\varphi^2} + \frac{1}{2}n^2\sin^{-1}\frac{\varphi}{n},$$

it is merely a matter of simple reductions to change equation (2) to the following form,

$$A_1 = 2\pi b\left[b + \frac{a^2}{\sqrt{a^2-b^2}}\sin^{-1}\left(\frac{\sqrt{a^2-b^2}}{a}\right)\right]. \tag{3}$$

Now, by the definition of $e$, $b^2 = a^2(1-e^2)$; so that, by elimination of $b$, equation (3) may be expressed as the following function of $a$ and $e$:

$$A_1 = 2\pi a^2(e^{-1}\sqrt{1-e^2}\sin^{-1}e + 1 - e^2). \tag{4}$$

*Case II.* Oblate spheroid.

Proceeding as above, it will be found that

$$A_2 = \frac{4\pi a}{b^2}\int_0^b dy\sqrt{(a^2-b^2)y^2+b^4}.$$

The following well-known relation

$$\int d\varphi\sqrt{\varphi^2+n^2} = \tfrac{1}{2}\varphi\sqrt{\varphi^2+n^2} + \tfrac{1}{2}n^2\log\left(\varphi+\sqrt{\varphi^2+n^2}\right)$$

facilitates the reduction of the preceding integral to

$$A_2 = 2\pi a\left[a + \frac{b^2}{\sqrt{a^2-b^2}}\log\left(\frac{a+\sqrt{a^2-b^2}}{b}\right)\right],$$

or

$$A_2 = 2\pi a^2\left[1 + \frac{1}{2}e^{-1}(1-e^2)\log\left(\frac{1+e}{1-e}\right)\right]. \tag{5}$$

*Remarks.* Both of the proposed expressions are incorrect. When $e = 1$ ($b = 0$) the given formula for the prolate spheroid leads to $2\pi a^2$ instead of zero. Again, when $e = 0$ ($b = a$) the series

$$\frac{1}{2}e^{-1}\log\left(\frac{1+e}{1-e}\right) = 1 + \frac{1}{3}e^2 + \frac{1}{5}e^4 + \cdots$$

shows that the given expression for the oblate spheroid reduces to $8\pi a^2$ instead of $4\pi a^2$ (sphere). Formulas (4) and (5) are equivalent to those given in Williamson's *Integral Calculus*, page 258.

*Note.* Contributors named below observed that the results as corrected are given in Lamb's *Infinitesimal Calculus,* 1902, page 273 and in Czuber's *Integralrechnung,* page 284. It is also given in *The Encyclopaedia Britannica,* ninth edition, volume 13, page 55.—EDITORS.

Also solved by NORMAN ANNING, J. A. BULLARD, NATHAN DEUTSCHMAN, A. FAYDER, L. G. FEMAN, MAURICE KRAUT, BENJAMIN LEVINE, I. H. MARANTZ, MOSES NISSENBAUM, H. L. OLSON, ARTHUR PELLETIER, H. W. REDDICK, H. A. RHODES, J. L. RILEY, D. H. RICHERT, J. B. REYNOLDS, and T. R. THOMSON.

**2815 [1920, 134]. Proposed by the late L. G. WELD.**

A right circular cone is laid upon an inclined plane so that its element of contact makes a given angle with the slant line of the plane. Assuming that there is no slipping and that the rolling friction is negligible, find the time of oscillation of the cone.

## I. SOLUTION BY J. B. REYNOLDS, Lehigh University.

The center of gravity of the cone moves in a circle of radius $\frac{3}{4}h\cos\alpha$ at a perpendicular distance $\frac{3}{4}h\sin\alpha$ from the plane, $h$ being the height of the cone and $2\alpha$ its vertical angle. When the radius of this circle makes an angle $\theta$ with the line of slope of the plane, the center of gravity of the cone has been raised a height $\frac{3}{4}h\cos\alpha(1-\cos\theta)\sin\beta$ above its lowest position, $\beta$ being the angle of slope of the plane. Then the potential energy, *P.E.*, is $\frac{3}{4}mgh\cos\alpha\sin\beta(1-\cos\theta)$, $m$ being the mass of the cone.

Since the line of contact is the instantaneous axis of rotation, the kinetic energy, *K.E.*, of the cone is $\frac{1}{2}I\dot{\varphi}^2$, $I$ being the moment of inertia with respect to the instantaneous axis or an element of the cone and $\dot{\varphi}$ the angular velocity about this axis.

Now the moment of inertia of a thin disc of radius $a$ and thickness $t$ with respect to a line through its center making an angle $\alpha$ with a perpendicular to its plane is

$$t\int_0^{2\pi}\int_0^a r^2(\cos^2\theta\cos^2\alpha + \sin^2\theta)r\,dr\,d\theta = \frac{\pi a^4}{4}(1+\cos^2\alpha)t$$

and for the moment of inertia of the disc about a parallel line at a distance $a\cos\alpha$, we have

$$\frac{\pi a^4}{4}(1+\cos^2\alpha)t + \pi a^2 t(a\cos\alpha)^2 = \frac{\pi a^4}{4}(1+5\cos^2\alpha)t.$$

So that, since the radius of any circular differential disc at distance $x$ from the vertex of the cone